COMBINATORICA 6 (1) (1986) 73—79

ON THE NUMBER OF PATHS AND CYCLES FOR ALMOST ALL GRAPHS AND DIGRAPHS

I. TOMESCU

Received 26 March 1983
Revised 24 April 1984

In this paper it is deduced the number of s-paths (s-cycles) having k edges in common with a fixed s-path (s-cycle) of the complete graph K_n (or K_n^* for directed graphs).

It is also proved that the number of the common edges of two s-path (s-cycles) randomly chosen from the set of s-paths (s-cycles) of K_n (respectively K_n^*), are random variables, distributed asymptotically in accordance with the Poisson law whenever $\lim_{n\to\infty} s/n$ exists, thus extending a result by Baróti.

Some estimations of the numbers of paths and cycles for almost all graphs and digraphs are made by applying Chebyshev's inequality.

1. Definition and notations

Throughout this paper we shall only be concerned with graphs and digraphs G of order n. If a path (cycle) of G has its length equal to s it will be called an s-path (s-cycle) of G.

We shall use the following notations:

$P_{n,s}(m)$—the number of graphs having exactly m $(s-1)$-paths;

$\xi_{n,s}$—the random variable taking the value m with the probability $P_{n,s}(m)/2^{\binom{n}{2}}$;

$C_{n,s}(m)$—the number of graphs having m s-cycles;

$\eta_{n,s}$—the random variable taking the value m with probability $C_{n,s}(m)/2^{\binom{n}{2}}$;

$DP_{n,s}(m)$—the number of digraphs having $m\,(s-1)$-paths;

$\mu_{n,s}$—the random variable taking the value m with probability $DP_{n,s}(m)/2^{n^2-n}$;

$DC_{n,s}(m)$—the number of digraphs having m s-cycles;

$\nu_{n,s}$—the random variable taking the value m with probability $DC_{n,s}(m)/2^{n^2-n}$;

$P(n, s, k)$, $C(n, s, k)$, $DP(n, s, k)$, $DC(n, s, k)$—the number of $(s-1)$-paths (s-cycles) having k edges in common with a given $(s-1)$-path (s-cycle) in the complete graph K_n (respectively complete digraph K_n^* having n^2-n arcs);

$M\xi_{n,s}$—the mathematical expectation of $\xi_{n,s}$;

$D\xi_{n,s}$—the dispersion of $\xi_{n,s}$;

$(x)_h=x(x-1)\dots(x-h+1)$ for any real x and natural h.

AMS subject classification (1980): 05 C 30

2. The case of graphs

The mean values $M\xi_{n,s}$ and $M\eta_{n,s}$ follow immediately from the definitions.

Proposition 1. *We have*

$$M\xi_{n,s}=\frac{(n)_s}{2^s} \quad\text{and}\quad M\eta_{n,s}=\frac{(n)_s}{s2^{s+1}}. \quad ▮$$

Baróti [1] obtained the number of Hamiltonian cycles of a complete graph K_n having r edges in common with a fixed Hamiltonian cycle C of K_n. An auxiliary result derived by Baróti was the following: If the edges of C are $e_1, \dots, e_n$, then the number of k-combinations of these edges that determine exactly j components on C is equal to

$$C_j(n,k)=\frac{n}{j}\binom{k-1}{j-1}\binom{n-k-1}{j-1}$$

for any $1\leqq j\leqq k$.

We shall deduce the number $P_j(n,k)$ of the selections of k edges from the set $\{e_1, \dots, e_{n-1}\}$ of the edges of a path P of length $n-1$, such that these k edges generate exactly j connected components on P.

Proposition 2. *The following relation holds:*

$$P_j(n,k)=\binom{k-1}{j-1}\binom{n-k}{j}.$$

Proof. If P is the path $x_1, x_2, \dots, x_n$, let Q: $x_0, x_1, \dots, x_n, x_{n+1}$ and the additional edges $e_0=x_0x_1$ and $e_n=x_nx_{n+1}$. $P_j(n,k)$ is equal to the number of the selections of k edges of Q which are different from e_0 and e_n and induce j components on Q. But this number is equal to the number of solutions of the system:

$$\begin{aligned} a_1+\dots+a_j &= k \\ b_1+\dots+b_{j+1} &= n-k+1 \end{aligned}$$

where a_i, b_i are integers and $a_i, b_i \geqq 1$, i.e. to $\binom{k-1}{j-1}\binom{n-k}{j}$. ▮

Now we are able to find, using Jordan's sieve formula, the expressions for $P(n,s,k)$ and $C(n,s,k)$.

Proposition 3. *We have*

$$P(n,s,k)=\sum_{i=k}^{s-1}(-1)^{i-k}\binom{i}{k}\frac{(s-i)!}{2}\sum_{j=1}^{i}\binom{i-1}{j-1}\binom{s-i}{j}\binom{n-i-j}{s-i-j}2^j.$$

Proof. Let P: $x_1, x_2, \dots, x_s$ be a fixed path of length $s-1$ of K_n and denote its edges by $e_1=x_1x_2, e_2=x_2x_3, \dots, e_{s-1}=x_{s-1}x_s$. Let A_i denote the set of all $(s-1)$-paths of K_n containing edge e_i of P for $1\leqq i\leqq s-1$. Then $P(n,s,k)$ is the number of $(s-1)$-paths of K_n which belong to precisely k sets A_i. By Jordan's sieve formula we obtain

$$P(n,s,k)=\sum_{i=k}^{s-1}(-1)^{i-k}\binom{i}{k}\sum_{\substack{K\subset\{1,\dots,s-1\}\\ |K|=i}}\Big|\bigcap_{p\in K}A_p\Big|.$$

If $K\subset\{1, \dots, s-1\}$ and $|K|=i$, suppose that the set of edges $E_K=\{e_p\}_{p\in K}$ generates exactly j components $P_1, \dots, P_j$ on P. In this case by applying the method in [1] one can show that

$$\Big|\bigcap_{p\in K} A_p\Big| = \frac{(s-i)!}{2}\binom{n-i-j}{s-i-j}2^j$$

for any $1\leqq j\leqq i\leqq s-2$. Now the result follows by virtue of the expression of $P_j(s, i)$. ∎

Proposition 4. *The following equality holds:*

$$C(n, s, k) = \sum_{i=k}^{s-1}(-1)^{i-k}\binom{i}{k}\frac{(s-i-1)!}{2}\sum_{j=1}^{i}\frac{s}{j}\binom{i-1}{j-1}\binom{s-i-1}{j-1}\binom{n-i-j}{s-i-j}2^j+$$
$$+(-1)^{s-k}\binom{s}{k}.$$

Proof. If C: $x_1, x_2, \dots, x_s, x_1$ is a fixed s-cycle of K_n and $E(C)=\{e_1, \dots, e_n\}$, let A_i denote the set of all s-cycles of K_n containing edge e_i of C for $1\leqq i\leqq s$. One can show, as in the case of $(s-1)$-paths, that

$$\Big|\bigcap_{p\in K} A_p\Big| = \frac{(s-i-1)!}{2}\binom{n-i-j}{s-i-j}2^j$$

for any $1\leqq j\leqq i\leqq s-3$, if the edge set $\{e_p\}_{p\in K}$ induces j components on C. If $|K|=s$, $s-1$ or $s-2$ then a simple counting argument implies that $\sum_{K\subset\{1,\dots,s\}}\Big|\bigcap_{p\in K} A_p\Big|=1$, s, and $s(n-s+1)+s(s-3)$, respectively.

Thus the proposition follows by applying Jordan's formula, if one considers that $\binom{0}{0}=1$. ∎

Proposition 5. *For any fixed k, if $\lim_{n\to\infty}\frac{s}{n}=\lambda\in[0, 1]$ we have:*

$$\lim_{n\to\infty}\frac{2P(n, s, k)}{(n)_s}=\lim_{n\to\infty}\frac{2sC(n, s, k)}{(n)_s}=\frac{2^k\lambda^{2k}}{k!}e^{-2\lambda^2}.$$

Proof. Denote $S(n, s, i, j)=\frac{(s-i)!}{2}\binom{i-1}{j-1}\binom{s-i}{j}\binom{n-i-j}{s-i-j}2^j$ for any fixed i and $1\leqq j\leqq i$. One can see that for $1\leqq j\leqq i-1$ we have $\lim_{n\to\infty}\frac{2S(n,s,i,j)}{(n)_s}=0$ and for $j=i$ we deduce that $\lim_{n\to\infty}\frac{2S(n,s,i,j)}{(n)_s}=\frac{2^i}{i!}\lambda^{2i}$. From the general theory of the principle of inclusion-exclusion (see e.g. Lovász' book [2]) it is known that the numbers $P(n, s, k)$ satisfy the Bonferroni inequalities, whence by routine application of the method in [1] it follows that

$$\lim_{n\to\infty}\frac{2P(n, s, k)}{(n)_s}=\sum_{p=0}^{\infty}(-1)^p\binom{p+k}{k}\frac{2^{p+k}}{(p+k)!}\lambda^{2(p+k)}=\frac{2^k\lambda^{2k}}{k!}e^{-2\lambda^2}.$$

In a similar way we find that $\lim\limits_{n\to\infty}\dfrac{2sC(n,s,k)}{(n)_s}=\dfrac{(2\lambda^2)^k}{k!}e^{-2\lambda^2}$. For $s=n$, hence $\lambda=1$, this result has been obtained by Baróti [1]. Since $(n)_s/2$ is the number of $(s-1)$-paths of K_n and $(2s)^{-1}(n)_s$ represents the number of s-cycles of K_n, it follows that the number of the common edges of two $(s-1)$-paths (s-cycles) randomly chosen from the set of all $(s-1)$-paths (s-cycles) of K_n are two random variables which are distributed asymptotically in accordance with the Poisson law whenever $\lim\limits_{n\to\infty}\dfrac{s}{n}$ exists. ▮

Proposition 6. *The following inequalities hold:*

$$D\xi_{n,s}<\frac{(s-1)^2}{n(n-1)}(e^4-e^2)(M\xi_{n,s})^2$$

and

$$D\eta_{n,s}<\frac{s^2}{n(n-1)}(e^4-e^2)(M\eta_{n,s})^2.$$

Proof. It is clear that

$$P(n,s,k)\leqq\frac{(s-k)!}{2}\sum_{j=1}^{k}\binom{k-1}{j-1}\binom{s-k}{j}\binom{n-k-j}{s-k-j}2^j\leqq$$

$$\frac{s-k}{k}(s-k)!\binom{n-k-1}{n-s}2^{k-1}\sum_{j=1}^{k}\binom{k-1}{j-1}\binom{s-k-1}{j}=\frac{2^{k-1}}{k!}(s-k)^2(s-j-1)!\binom{n-k-1}{n-s}.$$

We can write:

$$M\xi_{n,s}^2=\frac{1}{2^{\binom{n}{2}}}\left[\left[\left(\frac{1}{2}(n)_s\right)^2-\frac{1}{2}(n)_s\sum_{k=1}^{s}P(n,s,k)\right]2^{\binom{n}{2}-2s+2}+\right.$$

$$\left.+\frac{1}{2}(n)_s\sum_{k=1}^{s}P(n,s,k)2^{\binom{n}{2}-2s+k+2}\right]=$$

$$=(M\xi_{n,s})^2+(n)_s2^{-2s+1}\sum_{k=1}^{s}P(n,s,k)(2^k-1),$$

hence

$$D\xi_{n,s}=M\xi_{n,s}^2-(M\xi_{n,s})^2=(n)_s2^{-2s+1}\sum_{k=1}^{s}P(n,s,k)(2^k-1).$$

It follows that

$$\frac{D\xi_{n,s}}{(M\xi_{n,s})^2}=\frac{2}{(n)_s}\sum_{k=1}^{s}P(n,s,k)(2^k-1)<\frac{(s-2)!}{(n)_s}(s-1)^2\binom{n-2}{n-s}\sum_{k=0}^{\infty}\frac{2^k(2^k-1)}{k!}=$$

$$=\frac{(s-1)^2}{n(n-1)}(e^4-e^2).$$

In a similar way one can show that

$$C(n, s, k) < \frac{2^{k-1}}{k!} s(s-2)! \binom{n-k-1}{n-s},$$

which implies that

$$\frac{D\eta_{n,s}}{(M\eta_{n,s})^2} < \frac{s^2}{n(n-1)} (e^4 - e^2). \quad \blacksquare$$

Theorem 1. *For almost all graphs G of order n the number of $(s-1)$-paths of G, denoted by $P_s(G)$ and the number of s-cycles of G, denoted by $C_s(G)$ satisfy:*

$$\frac{(n)_s}{2^s}\left(1 - \frac{(s-1)\varphi(n)}{n}\right) < P_s(G) < \frac{(n)_s}{2^s}\left(1 + \frac{(s-1)\varphi(n)}{n}\right)$$

$$\frac{(n)_s}{s2^{s+1}}\left(1 - \frac{s\varphi(n)}{n}\right) < C_s(G) < \frac{(n)_s}{s2^{s+1}}\left(1 + \frac{s\varphi(n)}{n}\right)$$

as $n \to \infty$, where φ is any function such that $\lim_{n\to\infty} \varphi(n) = \infty$.

Proof. By the Chebyshev inequality we have for any $t > 0$:

$$P(|\xi_{n,s} - M\xi_{n,s}| \geqq t) \leqq \frac{D\xi_{n,s}}{t^2}.$$

In particular, if

$$t = \frac{(s-1)\varphi(n)}{n} M\xi_{n,s},$$

one can see that

$$\frac{D\xi_{n,s}}{t^2} \to 0$$

as $n \to \infty$, hence when $n \to \infty$, for almost all graphs G we have

$$\left|P_s(G) - \frac{(n)_s}{2^s}\right| < \frac{(s-1)\varphi(n)}{n} \cdot \frac{(n)_s}{2^s}.$$

A similar conclusion holds for $C_s(G)$. $\blacksquare$

Corollary 1.1. *For almost all graphs G the number of $(s-1)$-paths is equal to $\frac{n^s}{2^s} e^{-s^2/2n}(1+o(1))$ and the number of s-cycles is equal to $\frac{n^s}{s2^{s+1}} e^{-s^2/2n}(1+o(1))$ as $n \to \infty$, whenever $s = o(n^{2/3})$.*

Proof. It is well known that

$$\frac{(n)_s}{n^s} = e^{-s^2/2n}(1+o(1))$$

for $s = o(n^{2/3})$ (see e.g. [3]). For $\varphi(n) = n^{1/3}$ the result follows. $\blacksquare$

Note that Theorem 1 gives us only non trivial upper bounds for the number of Hamiltonian paths and cycles for almost all graphs G as $n \to \infty$.

3. The case of digraphs

We shall deduce that with slight modifications all results for graphs are valid also for digraphs.

Proposition 7. *The following equalities hold:*

$$M\mu_{n,s} = \frac{(n)_s}{2^{s-1}} \quad and \quad M\nu_{n,s} = \frac{(n)_s}{s2^s}. \quad \blacksquare$$

Proposition 8. *The numbers $DP(n, s, k)$ are given by*

$$DP(n, s, k) = \sum_{i=k}^{s-1} (-1)^{i-k} \binom{i}{k} (s-i)! \sum_{j=1}^{i} \binom{i-1}{j-1} \binom{s-i}{j} \binom{n-i-j}{s-i-j}. \quad \blacksquare$$

The proof is similar to that of the Proposition 3, since K^*_{s-i} has $(s-i)!$ Hamiltonian paths, and each such $(s-i-1)$-path may be expanded to a $(s-1)$-path of K^*_n in a unique way.

Proposition 9. *We have*

$$DC(n, s, k) = \sum_{i=k}^{s-1} (-1)^{i-k} \binom{i}{k} (s-i-1)! \sum_{j=1}^{i} \frac{s}{j} \binom{i-1}{j-1} \binom{s-i-1}{j-1} \binom{n-i-j}{s-i-j} + (-1)^{s-k} \binom{s}{k}.$$

Proof. If C is an s-cycle of K^*_n, denote by A_i the set of all s-cycles of K^*_n using the arc a_i of C for $1 \leqq i \leqq s$. Because K^*_{s-i} has $(s-i-1)!$ Hamiltonian cycles for $s-i \geqq 3$ and each $(s-i)$-cycle obtained in this way may be extended to an s-cycle of K^*_n in a unique way, the proof follows as for Proposition 4. $\blacksquare$

Note that for $i=s$, $s-1$, and $s-2$ the corresponding terms from the right-hand side of this equation are equal to those given by Jordan's formula, if we define $\binom{0}{0}=1$. Indeed, if $|K|=s$, $s-1$ or $s-2$ then $\sum_K \left|\bigcap_{p \in K} A_p\right| = 1$, s, and $s(n-s+1)+ +s(s-3)/2$, respectively. For the number of Hamiltonian cycles of K^*_n having exactly k arcs in common with a fixed Hamiltonian cycle, one obtains a simpler expression, namely:

$$DC(n, n, k) = \sum_{i=k}^{n-1} (-1)^{i-k} \binom{i}{k} \binom{n}{i} (n-i-1)! + (-1)^{n-k} \binom{n}{k},$$

since $\sum_{j=1}^{i} \frac{n}{j} \binom{i-1}{j-1} \binom{n-i-1}{j-1} = \binom{n}{i}$.

Proposition 10. *For every fixed $k \geqq 0$, if $\lim_{n\to\infty} \frac{s}{n} = \lambda$, the following equalities hold:*

$$\lim_{n\to\infty} \frac{DP(n, s, k)}{(n)_s} = \lim_{n\to\infty} \frac{sDC(n, s, k)}{(n)_s} = \frac{\lambda^{2k}}{k!} e^{-\lambda^2}. \quad \blacksquare$$

The proof is similar to that of the Proposition 5. It follows that the number of the common arcs of two $(s-1)$-paths, respectively s-cycles chosen at random from the set of all $(s-1)$-paths (s-cycles) of K^*_n are random variables which are distributed asymptotically also in accordance with the Poisson law if one supposes that there exists $\lim (s/n)$ as $n \to \infty$.

Proposition 11. *We have:*

$$D\mu_{n,s} < \frac{(s-1)^2}{n(n-1)}(2e^2-e)(M\mu_{n,s})^2,$$

$$D\nu_{n,s} < \frac{s^2}{n(n-1)}(2e^2-e)(M\nu_{n,s})^2.$$

Proof. One can show, as for undirected graphs, that $DP(n,s,k) < \frac{(s-k)^2(s-2)!}{(k+1)!}\binom{n-k-1}{n-s}$ and $DC(n,s,k) < \frac{s(s-2)!}{(k-1)!}\binom{n-k-1}{n-s}$. The result follows in the same way as for Proposition 6.

By applying Chebyshev's inequality we obtain the following consequences:

Theorem 2. *For almost all digraphs G of order n the number of $(s-1)$-paths of G, denoted by $DP_s(G)$ and the number of s-cycles of G, denoted by $DC_s(G)$ verify the estimations:*

$$\frac{(n)_s}{2^{s-1}}\left(1-\frac{(s-1)\varphi(n)}{n}\right) < DP_s(G) < \frac{(n)_s}{2^{s-1}}\left(1+\frac{(s-1)\varphi(n)}{n}\right),$$

$$\frac{(n)_s}{s2^s}\left(1-\frac{s\varphi(n)}{n}\right) < DC_s(G) < \frac{(n)_s}{s2^s}\left(1+\frac{s\varphi(n)}{n}\right)$$

as $n\to\infty$, where φ is any function having the property that $\lim_{n\to\infty}\varphi(n)=\infty$. ∎

Corollary 2.1. *When $n\to\infty$ for almost all digraphs G the number of $(s-1)$-paths is equal to $\frac{n^s}{2^{s-1}}e^{-s^2/2n}(1+o(1))$ and the number of s-cycles is equal to $\frac{n^s}{s2^s}e^{-s^2/2n}(1+o(1))$ whenever $s=o(n^{2/3})$.* ∎

From these estimations we can derive upper bounds for the number of Hamiltonian paths and cycles for almost all digraphs of order n when $n\to\infty$. It can be proved also that for any fixed s, when $n\to\infty$ almost all graphs and digraphs contain paths of any length p such that $p\leqq s$ and cycles of any length q if $3\leqq q\leqq s$.

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Ioan Tomescu

Facultatea de Matematică, Universitatea din Bucureşti,
Str. Academiei nr. 14,
R—70 109 Bucharest, Romania